A BOUNDARY-VALUE PROBLEM FOR A NONLINEAR HEAT-CONDUCTION EQUATION

K. G. Valeev and Yu. V. Rakitskii

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 5, pp. 52–61, 1967

Methods for the numerical solution of the boundary-value problem are considered. One method is related to the method of successive approximations and the other employs the collocation method [1]. A relationship between the latter method and the Ritz and Galerkin methods [2] is shown. An application of the collocation method to the nonstationary problem is given. The approximate solution is represented in analytic form. A way of finding the absolute error of the approximate solution is given.

1. Some inequalities related to the solution of the boundary-value problem. We consider the nonlinear partial differential equation in dimensionless form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial u^2} + \frac{\partial^2 u}{\partial y^2} - F(u, x, y). \tag{1.1}$$

The solution is sought in the simply connected region D with the boundary Γ with the given initial and boundary conditions

$$u|_{\Gamma} = 0, \qquad u|_{t=0} = \varphi(x, y), \qquad (x, y) \in D. \tag{1.2}$$

The nonlinear continuous function $F(u, x, y)$ is assumed to be increasing in u when x and y are fixed. It characterizes the heat losses due to heat transfer to the ambient medium.

First, let us consider the stationary problem

$$\Delta u = F(u, x, y), \quad (x, y) \in D, \qquad u|_{\Gamma} = 0 \tag{1.3}$$

and note certain inequalities. From physical considerations, the existence and uniqueness of the solution of boundary-value problem (1.3) are obvious.

Lemma 1. If $F(0, x, y) \leq 0$ when $(x, y) \in D$, then $u(x, y) \geq 0$ when $(x, y) \in D$.

Proof. Conversely, let the inequality

$$u(x, y) < 0, \qquad (x, y) \in D^*;$$
$$\text{and } (x, y) = 0, \qquad (x, y) \in \Gamma^* \tag{1.4}$$

be satisfied in some region $D^*(D^* \subset D)$ with the boundary Γ^*.

From Green's theorem

$$\int_{\Gamma^*} v \frac{\partial u}{\partial \nu} ds = \iint_{D^*} \left[\frac{\partial}{\partial x}\left(v \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial y}\right)\right] dx\, dy. \tag{1.5}$$

We find at $v \equiv u$

$$\iint_{D^*} [uF(u, x, y) + (\operatorname{grad} u)^2]\, dx\, dy = 0. \tag{1.6}$$

In view of the assumption that

$$F_u'(u, x, y) \geq 0, \qquad (x, y) \in D \tag{1.7}$$

we have the inequality

$$\iint_{D^*} uF(u, x, y)\, dx\, dy \geq 0. \tag{1.8}$$

This contradicts (1.6), since $u \not\equiv 0$ in D^*, and, therefore,

$$\iint_{D^*} (\operatorname{grad} u)^2\, dx\, dy > 0. \tag{1.9}$$

The lemma is proved. Its physical meaning is obvious. If heat influx occurs when $u = 0$, then in D the temperature u is nonnegative for the stationary solution.

Note 1. We can prove the following: if $F(0, x, y) < 0$ when $(x, y) \in D$ and $F_u'(u, x, y) \geq 0$, then $u(x, y) > 0$ when $(x, y) \in D$.

Now let us compare the solutions of (1.3) with different right-hand sides. Along with (1.3), we have the solution of the boundary-value problem

$$\Delta v = \Phi(v, x, y), \quad (x, y) \in D, \quad v|_{\Gamma} = 0. \tag{1.10}$$

Lemma 2. If in D

$$\Phi(v, x, y) \geqslant F(u, x, y) \text{ when } v \geqslant u, \tag{1.11}$$

then

$$u(x, y) \geqslant v(x, y) \text{ when } (x, y) \in D. \tag{1.12}$$

Proof. We shall assume the opposite. In some region $D^* \subset D$ with the boundary Γ^* let

$$u(x, y) < v(x, y), \quad (x, y) \in D^*,$$
$$u(x, y) = v(x, y), \quad (x, y) \in \Gamma^*. \tag{1.13}$$

If into (1.5) we substitute $u - v$ for v and $u - v$ for u, then from (1.3) and (1.10) we obtain

$$\iint_{D^*} (u - v)(F(u, x, y) - \Phi(v, x, y))\, dx\, dy < 0. \tag{1.14}$$

This contradicts assumption (1.13) and condition (1.11). The lemma is proved.

The physical meaning of the lemma is that when the heat outflow is greater, the temperature v in the stationary solution is lower.

Note 2. We can prove that when

$$\Phi(v, x, y) > F(u, x, y), \qquad (x, y) \in D \quad \text{when } v \geqslant u \tag{1.15}$$

we have the absolute inequality $u(x, y) > v(x, y)$, $(x, y) \in D$. These inequalities are similar in meaning to the maximum principle [3].

2. Finding the error of the approximate solution. Let the approximate solution $u = u_0(x, y)$ of problem (1.3), which satisfies the boundary condition

$$u_0(x, y) = 0, \qquad (x, y) \in \Gamma \tag{2.1}$$

be known.

We estimate the value

$$z = u - u_0. \tag{2.2}$$

We introduce the residue function of the approximate solution

$$\delta(x, y) = -\Delta u_0(x, y) + F(u_0, x, y). \tag{2.3}$$

The equations for z take the form

$$\Delta z = F(z + u_0, x, y) - \Delta u_0(x, y),$$
$$z|_\Gamma = 0. \quad (2.4)$$

The possibility of the representation

$$F(z + u_0, x, y) - \Delta u_0(x, y) =$$
$$= \delta(x, y) + \lambda(z, x, y)\, z, \quad (2.5)$$
$$\lambda(z, x, y) \geqslant 0, \qquad (x, y) \in D \quad (2.6)$$

follows from (1.7).

Let us introduce, together with $\delta(x,y)$, the auxiliary functions

$$\delta_1(x, y) = \min\{0, \delta(x, y)\}, \qquad \delta_1(x, y) \leqslant 0, \quad (2.7)$$
$$\delta_2(x, y) = \max\{0, \delta(x, y)\}, \qquad \delta_2(x, y) \geqslant 0, \quad (2.8)$$

and consider the boundary-value problems

$$\Delta z_1 = \delta_1(x, y) + \lambda(z_1, x, y)\, z_1, \qquad z_1|_\Gamma = 0, \quad (2.9)$$
$$\Delta z_2 = \delta_2(x, y) + \lambda(z_2, x, y)\, z_2, \qquad z_2|_\Gamma = 0. \quad (2.10)$$

From Lemma 2 and the obvious inequality $\delta_1(x,y) \leq \delta(x,y) \leq \delta_2(x,y)$, we have the bound for z

$$z_2 \leqslant z \leqslant z_1, \qquad (x, y) \in D. \quad (2.11)$$

It follows from (2.7), (2.8), and Lemma 1 that $z_1 \geq 0$, $z_2 \leq 0$. Using Lemma 2 and the positiveness of z_1, we can overestimate z_1 by discarding the positive term in (2.9). We find that

$$z_1(x, y) \leqslant w_1(x, y), \qquad (x, y) \in D, \quad (2.12)$$

where $w_1(x,y)$ is the solution of the boundary-value problem

$$\Delta w_1 = \min\{0, \delta(x, y)\}, \qquad w_1|_\Gamma = 0. \quad (2.13)$$

Similarly, we find the solution of the boundary-value problem

$$\Delta w_2 = \max\{0, \delta(x, y)\}, \qquad w_2|_\Gamma = 0 \quad (2.14)$$

and arrive at the following theorem.

Theorem 1. The error $z(x,y)$ (2.2) of the approximate solution $u_0(x,y)$ of boundary-value problem (1.3) is estimated by the inequality

$$w_2(x, y) \leqslant z(x,y) \leqslant w_1(x, y), \quad (x,y) \in D, \quad (2.15)$$

where w_1 and w_2 are the solutions of boundary-value problems (2.13) and (2.14), and the function $\delta(x,y)$ is defined in (2.3).

Corollary 1. The estimate can be simplified. We introduce the number M,

$$M \geqslant |\Delta u_0(x, y) - F(u_0(x, y), x, y)|,$$
$$(x, y) \in D + \Gamma, \quad (2.16)$$

and solve the boundary-value problem in the region D

$$\Delta w = -1, \qquad w|_\Gamma = 0. \quad (2.17)$$

The error of the solution of (1.3) is estimated by the formula

$$|u(x, y) - u_0(x, y)| \leqslant Mw(x, y). \quad (2.18)$$

Note 3. It follows from Theorem 1 that the smaller $|\delta(x,y)|$ in D, the smaller, in general, the error of the approximate solution $u_0(x,y)$. It is natural to think of using the collocation method, i.e., choosing $u_0(x,y)$ such that the residue $\delta(x,y)$ vanishes at given points (x_k, y_k) in D.

Note 4. Theorem 1 can be extended to the three-dimensional (n-dimensional) case. We shall formulate it without proof.

Theorem 2. We consider the boundary-value problem

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 + \partial^2 u / \partial z^2 - F(u, x, y, z) = 0,$$
$$(x, y, z) \in D \quad (2.19)$$

with the boundary condition

$$u(x, y, z) = 0, \qquad (x, y, z) \in \Gamma, \quad (2.20)$$

where D is a simply connected three-dimensional region with the boundary Γ.

The function F is continuous and does not decrease with increasing u:

$$F_u'(u, x, y, z) \geqslant 0, \ (x, y, z) \in D. \quad (2.21)$$

Let the approximate solution $u_0(x,y,z)$, which satisfies the boundary condition, be found. We find

$$M = \max|\Delta u_0(x, y, z) -$$
$$- F(u_0(x, y, z), x, y, z)|\, (x, y, z) \in D + \Gamma. \quad (2.22)$$

To estimate the approximate solution we have the inequality

$$|u(x, y, z) - u_0(x, y, z)| \leqslant Mw(x, y, z), \quad (2.23)$$

where $w(x,y,z)$ is the solution of the boundary-value problem

$$\Delta w + 1 = 0, \ w(x, y, z) = 0, \ (x, y, z) \in \Gamma. \quad (2.24)$$

Note 5. If the limits $u^{(1)}$ and $u^{(2)}$ of variation of u in the solution of (1.3) are known and between these u values we have

$$0 \leqslant \lambda_1 \leqslant F_u'(u, x, y) \leqslant \lambda_2, \ (x, y) \in D, \quad (2.25)$$

to find the error of the approximate solution we can, instead of the Poisson equations (2.13), (2.14), (2.17), and (2.24), solve the Helmholtz equation, which, in the case of (2.17) for example, takes the form $\Delta_w - \lambda_{1w} = 1$.

3. Finding the error of the solution in a unit circle. Let the region D be the interior of a unit circle. We seek the solution $u(\rho, \varphi)$ of the boundary-value problem

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = F(u, \rho, \varphi), \quad u(1, \varphi) = 0, \quad (3.1)$$

where the function $F(u, \rho, \varphi)$ is assumed to be increasing in u for fixed ρ and φ. Let the approximate solution

$$u = u_0(\rho, \varphi), \qquad u_0(1, \varphi) = 0, \quad (3.2)$$

be found somehow.

We calculate the maximum value of the residue at $0 \leq \rho \leq 1$, $0 \leq \varphi \leq 2\pi$

$$M = \max_{\rho,\varphi}\left|\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_0}{\partial\rho}\right) +\right.$$
$$\left.+ \frac{1}{\rho^2}\frac{\partial^2 u_0}{\partial\rho^2} - F(u_0, \rho, \varphi)\right|. \quad (3.3)$$

The error of the approximate solution is estimated by the formula

$$|u(\rho, \varphi) - u_0(\rho, \varphi)| \leqslant$$
$$\leqslant 0.25\,(1 - \rho^2)\, M \leqslant 0.25\, M. \quad (3.4)$$

This follows from the solution of auxiliary equation (2.17) and the use of Corollary 1.

4. **Method of successive approximations.** Stationary problem (1.3) can be solved by using a method of successive approximations that consists in the successive solution of a number of boundary-value problems for a Poisson equation of the form

$$\Delta u_{j+1} = F(u_j, x, y), \quad (x, y) \in D,$$
$$u_j|_\Gamma = 0 \qquad (j = 0, 1, 2, \ldots). \tag{4.1}$$

As u_0 we can take any function; for example, $u_0 \equiv 0$. It follows from Theorem 1 that the sequence $u_j(x, y)$ converges uniformly in D on the solution if

$$|w(x, y) F_u'(u, x, y)| < 1,$$
$$(x, y) \in D, \quad u^{(1)} \leqslant u \leqslant u^{(2)} \tag{4.2}$$

where $u^{(1)}$ and $u^{(2)}$ are the a priori known limits of variation of u. In particular, for the solution of (3.1) the convergence condition for the successive approximations takes the form

$$\max_{\rho, \varphi, u} |(1 - \rho^2) F_u'(u, \rho, \varphi)| < 4$$
$$(0 \leqslant \rho \leqslant 1, \; 0 \leqslant \varphi \leqslant 2\pi, \quad u^{(1)} \leqslant u < u^2). \tag{4.3}$$

To improve the convergence of $u_j(x, y)$, we can extract from $F(u, p, \varphi)$ the part that is linear in u, for example, using the condition

$$\min_\lambda \max_{\rho, \varphi, u} |F_u'(u, \rho, \varphi) - \lambda|, \tag{4.4}$$

where ρ, φ, and u are defined in (4.3). The successive approximations lead to the solution of a number of boundary-value problems for the Helmholtz equation

$$\Delta u_{j+1} - \lambda u_{j+1} = F(u_j, x, y) - \lambda u_j(x, y),$$
$$(x, y) \in D \;\; u_j|_\Gamma = 0, \qquad u_0 \equiv 0 \quad (j = 0, 1, 2, \ldots). \tag{4.5}$$

Example. If by this method we solve the boundary-value problem at $u \geq 0$

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = -1 + 0.1 \; u^2, \quad u = 0$$
$$(x^2 + y^2 = 1) \tag{4.6}$$

we obtain, using polar coordinates,

$$u_0 = 0, \quad u_1(x, y) = 0.25 - 0.25 (x^2 + y^2), \; u_2(x, y) =$$
$$= 0.2454 - 0.2438 (x^2 + y^2) - 0.0016 (x^2 + y^2)^2. \tag{4.7}$$

From estimate (3.4) we find that the error of the approximate solution $u_2(x, y)$ does not exceed 0.0003.

5. **Solution of the inhomogeneous Helmholtz equation.** Boundary-value problems for inhomogeneous linear equations must be solved in the method of successive approximations. For a circular region, for the Helmholtz equation it is often necessary to find $u(\rho, \varphi)$, where

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \varphi^2} - \lambda u = \sum_{k=0}^{\infty} f_k(\rho) \cos 2k\varphi,$$
$$u(1, \varphi) = 0, \tag{5.1}$$

$$f_k(\rho) = \sum_{n=0}^{\infty} f_{kn} \rho^{2(k+n)}. \tag{5.2}$$

We shall assume that the series in (5.1) and (5.2) converge absolutely and uniformly when $|\rho| \leq 1$, $0 \leq \varphi \leq 2\pi$. We seek the solution $u(\rho, \varphi)$ in the form of a series

$$u(\rho, \varphi) = \sum_{k=0}^{\infty} w_k(\rho) \cos 2k\varphi, \qquad w_k(1) = 0. \tag{5.3}$$

To find $w_k(\rho)$, we obtain the equations

$$\frac{1}{\rho} \frac{d}{d\rho}\left(\rho \frac{dw_k(\rho)}{d\rho}\right) - \frac{4k^2}{\rho^2} w_k(\rho) - \lambda w_k(\rho) = f_k(\rho). \tag{5.4}$$

This is an inhomogeneous Bessel equation. Direct calculations convinced the authors that it was more convenient to seek the solution $w_k(\rho)$ at once in the form of a special series

$$w_k(\rho) = \rho^{2k}\left[a_0 \frac{\rho^2 - 1}{2(4k + 2)} + \right.$$
$$\left. + a_1 \frac{\rho^4 - 1}{4(4k + 4)} + a_2 \frac{\rho^6 - 1}{6(4k + 6)} + \ldots\right]. \tag{5.5}$$

For the coefficients a_n we obtain the equations

$$a_0 + \lambda \sum_{n=0}^{\infty} \frac{1}{(2n + 2)(4k + 2n + 2)} a_n = f_{k_0}, \tag{5.6}$$

$$a_n - \lambda \left[\frac{1}{2n(4k + 2n)}\right] a_{n-1} = f_{kn} \qquad (n = 1, 2, \ldots). \tag{5.7}$$

When a_0 is undetermined, a_1, a_2, ... are determined successively from (5.7). If we substitute them into (5.6), we obtain a_0. With many iterations, it is first of all convenient to find partial solutions that correspond to a one-term right side of the form

$$f_k(\rho) = \rho^{2k+2n} \qquad (n = 0, 1, 2, \ldots).$$

6. **Estimates of nonstationary solutions.** Let $u(t, x, y)$ be the solution of Eq. (1.1).

Let us consider the integral

$$\Omega(t) = \iint_D u^2(t, x, y)\, dx\, dy. \tag{6.1}$$

From (1.10), differentiating (6.1), we obtain

$$\frac{d\Omega(t)}{dt} = 2 \iint_D u \Delta u dx\, dy - 2 \iint_D u F(u, x, y)\, dx dy. \tag{6.2}$$

Let A be the maximum value of w, where w is the solution of boundary-value problem (2.17),

$$\Delta w = -1, \qquad w|_\Gamma = 0,$$
$$A = \max_{x, y} w(x, y), \; (x, y) \in D. \tag{6.3}$$

The value A is a function only of the region D. From Note 1, we find that $w(x, y) > 0$ when $(x, y) \in D$.

Let us introduce the eigenvalues and eigenfunctions of the boundary-value problem ([1], p. 671)

$$\Delta u + \mu u = 0, \qquad u|_\Gamma = 0. \tag{6.4}$$

We shall let $v_1(x, y)$, $v_2(x, y)$, ... be the eigenfunctions.

The corresponding eigenvalues μ will be $\mu_1, \mu_2, \ldots$. In the numbering, it is assumed that

$$0 < \mu_1 \leqslant \mu_2 \leqslant \ldots . \qquad (6.5)$$

Using the extremal nature of the eigenvalues and eigenfunctions, we find that the minimum of the integral

$$-\iint_D u\Delta u \, dx\, dy = \iint_D \operatorname{grad}^2 u \, dx\, dy \qquad (6.6)$$

for the functions u, which are continuous in D along with the second derivatives and which satisfy the condition

$$\iint_D u^2 dx\, dy = 1, \qquad u|_\Gamma = 0 \qquad (6.7)$$

is reached at $u = v_1$ and has the value μ_1. We have

$$\mu_1 \iint_D u^2 dx\, dy \leqslant -\iint_D u\Delta u \, dx\, dy . \qquad (6.8)$$

From Theorem 1, when $\delta(x, y) = -1$ in (2.14), for the equation

$$\Delta v_1 + \mu_1 v_1 = 0 \qquad (6.9)$$

and condition (6.3) we obtain the estimate

$$v_1(x, y) \leqslant \max v_1(x, y)\, \mu_1 A_1 . \qquad (6.10)$$

Hence, we have the lower bound for the first eigenvalue μ_1

$$A^{-1} \leqslant \mu_1 . \qquad (6.11)$$

Let condition (2.25) be satisfied. When $u \geq 0$, we obtain the following inequality for any α:

$$uF(u, x, y) \geqslant \lambda u^2 + F(0, x, y)\, u \geqslant$$

$$\geqslant (\lambda - {}^1/_2\,\alpha^2)\, u^2 - {}^1/_2\, \alpha^{-2} F^2(0, x, y). \qquad (6.12)$$

From inequalities (6.8), (6.11), and (6.12), we arrive in (6.2) at the differential inequality

$$\frac{d\Omega(t)}{dt} \leqslant -\left(\frac{2}{A} + 2\lambda - \alpha^2\right)\Omega(t) +$$

$$+ \frac{1}{\alpha^2}\iint_D F^2(0, x, y)\, dx\, dy . \qquad (6.13)$$

If we integrate (6.13), we obtain the integral estimate of the solution

$$\iint_D u^2(t, x, y)\, dx\, dy \leqslant e^{-\varkappa t}\iint_D u^2(0, x, y)\, dx\, dy +$$

$$+ \frac{1 - e^{-\varkappa t}}{\varkappa\alpha^2}\iint_D F^2(0, x, y)\, dx\, dy,$$

$$\varkappa = 2A^{-1} + 2\lambda - \alpha^2 > 0 . \qquad (6.14)$$

Many different inequalities for the solution of Eq. (1.1) can be derived from formula (6.14). Let, for example, $u_0(x, y)$ be the stationary solution of problem (1.1) and (1.2). We shall let

$$z(t, x, y) = u(t, x, y) - u_0(x, y). \qquad (6.15)$$

For z, we have the equations

$$\partial z / \partial t = \Delta z - [F(z + u_0, x, y) - \Delta u_0],$$
$$z|_\Gamma = 0, \qquad z|_{t=0} = \varphi(x, y) - u_0(x, y). \qquad (6.16)$$

From (6.14) we obtain the inequality

$$\iint_D [u(t, x, y) - u_0(x, y)]^2\, dx\, dy \leqslant$$

$$\leqslant e^{-\varkappa t}\iint_D [\varphi(x, y) - u_0(x, y)]^2\, dx\, dy ,$$

$$\varkappa = 2A^{-1} + 2\lambda , \qquad (6.17)$$

where A is defined in (6.3) and λ in (2.25). Estimates can be obtained for the difference between solutions of problem (1.1) and (1.2) with different initial conditions. From Section 3, we find that for a unit circle $A^{-1} = 4$.

7. The Galerkin method in the nonstationary problem. We take the set of orthogonal normalized functions $\psi_j(x, y)$

$$\iint_D \psi_j(x, y)\, \psi_k(x, y)\, dx\, dy = 0 \qquad (j \neq k),$$

$$\iint_D \psi_j^2(x, y)\, dx\, dy = 1, \qquad \psi_j|_\Gamma = 0. \qquad (7.1)$$

We shall seek the approximate solution of problem (1.1)–(1.3) in the form of a finite sum

$$u^*(t, x, y) =$$

$$= a_1(t)\psi_1(x, y) + \ldots + a_n(t)\psi_n(x, y),$$

$$u^*|_\Gamma = 0 . \qquad (7.2)$$

By substituting $u^*(t, x, y)$ into (1.1), we obtain the residue

$$\delta(t, x, y) \equiv \partial u^* / \partial t - \Delta u^* + F(u^*, u, y). \qquad (7.3)$$

We select the coefficients $a_j(t)$ from the condition that $\delta(t, x, y)$ be orthogonal to the functions $\psi_j(x, y)$ $(j = 1, \ldots, n)$ in D. We obtain the system of ordinary differential equations

$$\frac{da_j}{dt} = \sum_{i=1}^{n} \beta_{ji} a_i - f_j(a_1, \ldots, a_n) \qquad (j = 1, \ldots, n) , \qquad (7.4)$$

$$\beta_{ij} = \iint_D \psi_j \Delta\psi_i \, dx\, dy, \qquad f_j = \iint_D F(u^*, x, y)\, \psi_j\, dx\, dy. \qquad (7.5)$$

The initial conditions for a_j are found, using (1.2), from the equations

$$a_j(0) = \iint_D \psi_j(x, y)\, \varphi(x, y)\, dx\, dy . \qquad (7.6)$$

If we solve (7.4), using analog computers, for example, we find the approximate solution $u^*(t, x, y)$. Formula (6.14) can be used to estimate the error. We shall show that system of differential equations (7.4) is asymptotically stable. First, we shall demonstrate the boundedness of the solutions. If we multiply the j-th equation of (7.4) by a_j and sum, we find

$$\sum_{j=1}^{n} a_j \frac{da_j}{dt} \iint_D u^* \Delta u^* \, dx\, dy - \iint_D F(u^*, x, y) u^* \, dx\, dy. \quad (7.7)$$

From inequalities (6.6), (6.12), and (7.7) we obtain

$$\frac{d}{dt} \sum_{j=1}^{n} a_j^2 \leqslant -(2A^{-1} + 2\lambda - \alpha^2) \iint_D (u^*)^2 \, dx\, dy + $$

$$+ \alpha^{-2} \iint_D F^2(0, x, y)\, dx\, dy\,, \quad (7.8)$$

where α is any number. We shall select it from the condition

$$\varkappa = 2A^{-1} + 2\lambda - \alpha^2 > 0. \quad (7.9)$$

Since we have

$$\iint_D (u^*)^2 \, dx\, dy = \iint_D \sum_{i,j=1}^{n} a_i a_j \psi_i \psi_j \, dx\, dy = \sum_{j=1}^{n} a_j^2, \quad (7.10)$$

from (7.8) we find the estimate for the approximate solution

$$\sum_{j=1}^{n} a_j^2 \leqslant e^{-\varkappa t} \iint [u^*(0, x, y)]^2 \, dx\, dy +$$

$$+ \frac{1 - e^{-\varkappa t}}{\varkappa \alpha^2} \iint_D F^2(0, x, y)\, dx\, dy. \quad (7.11)$$

Note that the inequality

$$\sum_{i,j=1}^{n} \beta_{ji} a_i a_j \leqslant -\mu_1 \sum_{j=1}^{n} a_j^2 \leqslant -\frac{1}{A} \sum_{j=1}^{n} a_j^2 \quad (7.12)$$

follows from (6.8) and (6.11).

Along with the solution $a_i(t)$ of system (7.4), let us consider another solution $b_i(t)$ whose initial values differ from (7.6). We have

$$\frac{db_j}{dt} = \sum_{i=1}^{n} \beta_{ji} b_i - f_j(b_1, \ldots, b_n) \quad (j = 1, \ldots, n). \quad (7.13)$$

From (7.4) and (7.13) we obtain

$$\frac{d}{dt} \sum_{j=1}^{n} (a_j - b_j)^2 = 2 \sum_{i,j=1}^{n} \beta_{ji} (a_j - b_j) \times$$

$$\times (a_i - b_i) - Q(a_1, \ldots, a_n, b_1, \ldots, b_n). \quad (7.14)$$

Here,

$$Q(a_1, \ldots, a_n, b_1, \ldots, b_n) =$$

$$= 2 \sum_{j=1}^{n} (a_j - b_j) [f_j(a_1, \ldots, a_n) - f_j(b_1, \ldots, b_n)] =$$

$$= 2 \iint_D (u^* - u^\circ)(F(u^*, x, y) - F(u^\circ, x, y))\, dx\, dy, \quad (7.15)$$

$$u^\circ = b_1(t) \psi_1(x, y) + \ldots + b_n(t) \psi_n(x, y). \quad (7.16)$$

In view of assumption (1.7), a nonnegative function is under the integral in (7.15). From (7.12) and (7.14), therefore, we have the differential inequality

$$\frac{d}{dt} \sum_{j=1}^{n} (a_j - b_j)^2 \leqslant -2A^{-1} \sum_{j=1}^{n} (a_j - b_j)^2. \quad (7.17)$$

Integrating (7.17), we find the estimate

$$\sum_{j=1}^{n} [a_j(t) - b_j(t)]^2 \leqslant$$

$$\leqslant \exp\{-2A^{-1} t\} \sum_{j=1}^{n} [a_j(0) - b_j(0)]^2, \; A > 0. \quad (7.18)$$

The existence of a unique stationary solution of system (7.4) follows from (7.11) and (7.18). This solution is uniformly asymptotically stable.

8. **Application of the collocation method to the nonstationary problem.** Let us consider some quadrature formula with nodes at the points $M_i(x_i, y_i)$ for calculation in D of the definite integral

$$\iint_D \delta(x, y)\, dx\, dy \approx \sum_{i=1}^{n} A_i \delta(x_i, y_i), \qquad M_i \in D. \quad (8.1)$$

Equations (7.4) are approximately represented as

$$\iint_D \delta(t, x, y) \psi_j(x, y)\, dx\, dy \approx$$

$$\approx \sum_{i=1}^{n} A_i \delta(t, x_i, y_i) \psi_j(x_i, y_i) = 0 \quad (j = 1, \ldots, n). \quad (8.2)$$

Equations (8.2) can, in turn, be satisfied if

$$\delta(t, x_i, y_i) \equiv \sum_{j=1}^{n} \left[\frac{da_j}{dt} \psi_j(x_i, y_i) - \right.$$

$$\left. - a_j \Delta \psi_j(x_i y_i) \right] + F(u_i^*, x_i, y_i) = 0, \quad (8.3)$$

$$u_i^* = a_1(t) \psi_1(x_i, y_i) + \ldots + a_n(t) \psi_n(x_i, y_i). \quad (8.4)$$

The main advantage of Eqs. (8.3) is the simplicity of their formulation. This advantage of the collocation method over the Galerkin method or Ritz method (for the stationary problem) becomes obvious with nonlinear equations. In this case, exact calculation of the integrals in (7.5) is often impossible. The use of quadrature formulas results in Eqs. (8.2), which are equivalent to the equations of the collocation method. If a formula of increased accuracy is used when the collocation points are the nodes of the quadrature formula, the results of the Galerkin and collocation methods become virtually equivalent.

Let quadrature formula (8.1) give an accurate answer in calculation of integrals of the form

$$\iint_D \psi_j(x, y) \psi_k(x, y)\, dx\, dy = \delta_{jk} \quad (8.5)$$

where δ_{jk} is the Kronecker symbol. If we take the eigenfunctions $v_j(x, y)$ of boundary-value problem (6.4) with different eigenvalues μ_j (6.5) for the normalized orthogonal functions $\psi_j(x, y)$, solution of Eqs. (8.3) with respect to the derivatives results in the equations

$$\frac{da_j}{dt} + \mu_j a_j + \\ + \sum_{i=1}^{n} A_i v_j(x_i, y_i) F\left(\sum_{k=1}^{n} a_k v_k(x_i, y_i), x_i, y_i\right) = 0. \quad (8.6)$$

Another course in the collocation method consists in selection of functions $\psi_j(x, y)$ that satisfy the conditions

$$\psi_j(x_i, y_i) = \delta_{ij}, \quad \delta_{ii} = 1, \\ \delta_{ij} = 0 \quad (i \neq j), \quad \psi_j|_\Gamma = 0. \quad (8.7)$$

Equations (8.3) take the form

$$\frac{da_i}{dt} = \sum_{j=1}^{n} a_j \Delta\psi_j(x_i, y_i) - F(a_i, x_i, y_i) \\ (i = 1, \ldots, n). \quad (8.8)$$

Note that the linear part will be simple in system (8.6), while in system (8.8) the nonlinear functions, which are functions only of one of the variables a_j, will be simple. System (8.8) has a unique uniformly asymptotically stable solution by virtue of its similarity to system (7.4).

Example. We solve the boundary-value problem

$$\partial u / \partial t = \partial^2 u / \partial x^2 + 1, \quad u|_{t=0} = 0, \\ u|_{x=0} = u|_{x=1} = 0. \quad (8.9)$$

Using only one collocation point x = 0.5, we shall seek the solution as the function

$$u(t, x) = a(t)\, x(1 - x), \quad a(0) = 0 \quad (8.10)$$

which satisfies the initial and boundary conditions.

If we substitute u(t,x) into (8.9) at x = 0.5, we have

$$0.25\, da(t)/dt = -2a(t) + 1, \quad a(0) = 0. \quad (8.11)$$

We find the approximate solution

$$u_0(t, x) = 0.5\,(x - x^2)(1 - e^{-8t}). \quad (8.12)$$

The exact solution has the form

$$u(t, x) = \frac{4}{\pi^3} \sum_{n=0}^{\infty} \sin(2n+1)\pi x \{1 - \\ - \exp[-(2n+1)^2 \pi^2 t]\}. \quad (8.13)$$

The maximum deviation of the approximate solution from the exact is 0.008 at t ≈ 0.12 and x = 0.5. The stationary solution is found exactly.

9. Solution of the stationary boundary-value problem. In seeking the stationary solution of problem (1.1) and (1.2) by the Ritz, Galerkin, or collocation method, we obtain a system of nonlinear equations. For example, with the coordinate functions $\psi_j(x, y)$, which satisfy conditions (8.7), the collocation method gives the equations

$$\sum_{j=1}^{n} a_j \Delta\psi_j(x_i, y_i) - F(a_i, x_i, y_i) = 0 \\ (i = 1, \ldots, n). \quad (9.1)$$

They can be solved by the method of successive approximations, which follows from system (8.8),

$$a_{i, m+1} = a_{im} + h\Big[\sum_{j=1}^{n} a_{jm} \Delta\psi_j(x_i, y_i) - \\ - F(a_{im}, x_i, y_i)\Big] \\ (i = 1, \ldots, n, \; m = 0, 1, 2, \ldots, h > 0). \quad (9.2)$$

We take arbitrary values for a_{i0}. For sufficiently small h > 0, by virtue of the asymptotic stability of the solutions of system (8.8), the solution a_i of system (9.1) is found from (9.2)

$$a_i = \lim a_{im} \text{ for } m \to \infty. \quad (9.3)$$

Note that for the nonstationary problem (1.1) and (1.2), the collocation method is much less time-consuming than the net method.

REFERENCES

1. L. Collatz, Numerical Methods for Solving Differential Equations [Russian translation], Izd. inostr. lit., 1953.

2. S. G. Mikhlin, Numerical Realization of Variational Methods [in Russian], Izd-vo Nauka, 1966.

3. L. Bers, F. John, and M. Schechter, Partial Differential Equations [Russian translation], Izd. Mir, 1966.

19 January 1967 Kiev and Leningrad